On the two differences $l_R(I^*/R) - l_R(R/I)$ and $rl_R(R/I) - l_R(I^*/R)$

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Abstract. Let R be a one-dimensional local Noetherian domain, which is supposed analytically irreducible and residually rational, and let I be a proper ideal of R. Our purpose is to study the two numbers

$$a(I) := l_R(I^*/R) - l_R(R/I)$$
$$b(I) := rl_R(R/I) - l_R(I^*/R)$$

already considered in the literature under various points of view. The basic idea is the expression of these invariants in terms of the type sequence.

1 Introduction.

Let $(R, \mathfrak{m})$ be a one-dimensional local Noetherian domain with residue field k and quotient field K, which is analytically irreducible and residually rational. We denote by:

$\overline{R}$	the	*normalization* of R, $\overline{R} = k[[t]]$;
ω	a	*canonical module* of R such that $R \subseteq \omega \subseteq \overline{R}$;
γ	$:=$	$R : \overline{R}$ the *conductor ideal* of R in $\overline{R}$;
c	$:=$	$l_R(\overline{R}/\gamma)$, so that $\gamma = t^c\overline{R}$;
δ	$:=$	$l_R(\overline{R}/R)$ the *singularity degree* of R;
n	$:=$	$c - \delta = l_R(R/\gamma)$;
r	$:=$	$l_R(R : \mathfrak{m}/R)$ the $Cohen-Macaulay\ type$ of R;
I^*	$:=$	$R :_K I$ the *dual* of the fractional ideal I;
θ_D	$:=$	ω^* the $Dedekind\ different$ of R.

Given any proper nonzero ideal I of R, we use the notion of *type sequence* (see 2.2) in order to get informations about the two numerical differences:

$$a(I) := l_R(I^*/R) - l_R(R/I)$$
$$b(I) := rl_R(R/I) - l_R(I^*/R).$$

Having in mind the Gorenstein case with the following well-known equivalent characterizations (see [3]; [10]; [11], Theorem 13.1):

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$$R \text{ is Gorenstein} \iff \omega = R \iff r = 1 \iff 2\delta - c = 0$$
$$\iff a(I) = 0 \ \ \text{for every nonzero proper ideal } I$$

we get similar characterizations for *almost Gorenstein* rings (see Theorem 3.14):

$$R \text{ is almost Gorenstein} \iff \mathfrak{m}\omega = \mathfrak{m} \iff r - 1 = 2\delta - c$$
$$\iff a(I) = r - 1 - l_R(I^{**}/I) \ \ \text{for every nonzero proper ideal } I.$$

In the general case a direct calculation gives immediately that

$$a(I) \leq 2\delta - c$$

for every nonzero ideal I. The close relation with the type sequence of R

$$2\delta - c = \sum_{h=1}^{n} (r_h - 1)$$

induces us to search which elements of this sequence contribute in our invariants. This is discussed in Section (4). First, in Theorem 3.10, we obtain the formulas:

$$(A) \qquad a(I) = \sum_{h \in \{1,\dots,n\} \setminus V^I} (r_h - 1) - \ l_R(I^{**}/I) \ - d(I)$$

$$(B) \qquad b(I) = \sum_{h \notin V^I} (r - r_h) + \ r l_R(I^{**}/I) + d(I)$$

where $V^I \subseteq \mathbb{N}$ is a subset in biunivocal correspondence with the valuations of the ideal I^{**} and $d(I)$ is a non-negative invariant (see 3.6), closely related to the type sequence $[r_1, \dots, r_n]$ of R and to the valuations of I^{**}.
Successively lower and upper bounds and vanishing conditions for the invariants $a(I)$ and $b(I)$ are derived directly from these expressions. For instance

$$(A_1) \qquad a(I) \leq (r-1) l_R\big(R/(I^{**} + \theta_D)\big) - \ l_R(I^{**}/I)$$

which improves the inequality $a(I) \leq (r-1) l_R(R/I)$ obtained by Jäger in [10], Korollar 3, (2) and

$$(A_2) \qquad a(I) \geq (r-1) - \ l_R(I^{**}/I) \ - d(I)$$

which gives a sufficient condition for the positivity of $a(I)$.
We recall that in [3], Anm.5, R. Berger conjectured that always $a(I) \geq 0$, but there are counterexamples, we cite the following, exhibited by Jäger in [10]:

if $R = k[[t^9, t^{15}, t^{17}, t^{23}, t^{25}, t^{29}, t^{31}]]$ and $I = (t^{38}, t^{44}, t^{50})$, then $a(I) = -1$.

From the preceding (A_2) it turns out that $a(I) \geq r - 1 \geq 0$ for every integrally closed ideal I, because this condition implies that $I = I^{**}$ and also that $d(I) = 0$. The same holds for every ideal I such that $\omega \subseteq I : I$.
If R is *almost Gorenstein*, then $a(I) = 2\delta - c \geq 0$ for every reflexive ideal I.

Formula (B), by giving $b(I)$ as a sum of non negative terms, provides the fact that always

$$b(I) \geq r \ l_R(I^{**}/I) \geq 0$$

and also the following vanishing condition:

$$(VC) \qquad b(I) = 0 \iff I^{**} = I, \ d(I) = 0, \ \ r_i = r \ \text{ for all } i \notin V^I$$

Since by definition

$$a(I) + b(I) = (r-1)\, l_R(R/I)$$

it is clear that inequalities (A_1) and (A_2) may be read respectively as a lower and an upper bound for $b(I)$. We explicit these for the convenience of the reader.

$$(B_1) \qquad b(I) \geq (r-1)l_R(I^{**}+\theta_D/I) + \ l_R(I^{**}/I)$$

$$(B_2) \qquad b(I) \leq (r-1)(l_R(R/I)-1) + l_R(I^{**}/I) + d(I)$$

In the literature more attention has been reserved to the particular case $I = \gamma$. Notice that

$$a(\gamma) = 2\delta - c \quad \text{and} \quad b(\gamma) = r(c-\delta) - \delta$$

As concerns the number $b(\gamma)$, in [14], Theorem 3.7, the lower bound

$$b(\gamma) \geq l_R(\theta_D/\gamma)(r-1)$$

and in [5], Proposition 2.1, the upper bound

$$b(\gamma) \leq (r-1)[l_R(R/\gamma) - 1]$$

are established. Hence results B_1 and B_2 may be viewed as an extension of these bounds to any ideal I.

There are few cases in which $b(\gamma) \leq r$ (see 4.7). A general structure theorem for rings satisfying the equality $b(\gamma) = 0$ or $b(\gamma) = 1$ is presented in [4]: these rings are called rings *of maximal* or *almost maximal length*, respectively. Note that for $I = \gamma$ the above condition (VC) becomes:

$$b(\gamma) = 0 \iff r_i = r \;\; for\ each \;\; i = 1, ..., n.$$

Indeed, the rings of maximal length are exactly those having constant type sequence.
In a series of recent papers (see [6], [7], [8]) the authors attack the problem of classifying rings according to the value of the quantity $b(\gamma)$. In the last section we show how type sequences are an useful instrument from this point of view, by obtaining a complete classification of all possible rings having $b(\gamma) \leq r$.

2 Preliminaries and notations.

Throughout this paper $(R, \mathfrak{m})$ denotes a one-dimensional local Noetherian domain with residue field k. For simplicity, we assume that k is an infinite field. Let $\overline{R}$ be the integral closure of R in its quotient field K; we suppose that $\overline{R}$ is a finite R-module and a DVR with a uniformizing parameter t, which means that R is analytically irreducible. We also suppose R to be residually rational, i.e., $k \simeq \overline{R}/t\overline{R}$. We denote the usual valuation associated to $\overline{R}$ by

$$v : K \longrightarrow \mathbb{Z} \cup \infty, \qquad v(t) = 1.$$

In particular $v(R) := \{v(a),\ a \in R,\ a \neq 0\} \subseteq \mathbb{N}$ is the *numerical semigroup* of R. Under our hypotheses, for any fractional ideals $I \supseteq J \neq (0)$ the length of the R-module I/J can be computed by means of valuations (see [12], Prop. 1):

$$l_R(I/J) = \#(v(I) \setminus v(J)).$$

Given two fractional ideals I, J we define $I : J = \{x \in K \mid xJ \subseteq I\}$.

2.1

In our hypotheses R has a *canonical module* ω, unique up to isomorphism. Once for all we assume that

$$R \subseteq \omega \subset \overline{R}$$

We shall use the following properties (see [9]):

(1) $\omega : \omega = R$ and $\omega : (\omega : I) = I$ for every fractional ideal I.

(2) $l_R(I/J) = l_R(\omega : J/\omega : I)$ for every fractional ideals $I \supseteq J$.

(3) R is Gorenstein *if and only if* $\omega = R$ *if and only if* $\theta_D = R$.
Otherwise $\gamma_R \subseteq \theta_D \subseteq \mathfrak{m}$.

(4) $v(\omega) = \{j \in \mathbb{Z} \mid c-1-j \notin v(R)\}$, hence $c-1 \notin v(\omega)$ and $c+\mathbb{N} \subseteq v(\omega)$.

(5) (see [14], Lemma 2.3). For every fractional ideal I,

$$s \in v(I\omega) \quad if\ and\ only\ if \quad c-1-s \notin v(R : I).$$

2.2

The notion of *type sequence* has been introduced by Matsuoka in 1971 and recently revisited in [1]; we recall its definition. Let $n := l_R(R/\gamma)$ and let

$$s_0 = 0 < s_1 < \ldots < s_n = c < s_{n+1} = c+1 < ...$$

be the elements of $v(R)$. For each $i \geq 1$, define the ideal

$$R_i := \{a \in R \mid v(a) \geq s_i\}.$$

The chain $R = R_0 \supset R_1 = \mathfrak{m} \supset R_2 \supset \ldots \supset R_n = \gamma_R \supset R_{n+1} \supset ...$
induces the chain of duals

$$R \subset R : R_1 \subset .... \subset R : R_n = \overline{R} \subset R : R_{n+1} = t^{-1}\overline{R} \subset ...$$

For every $i \geq 1$, we put

$$r_i := l_R(R : R_i/R : R_{i-1}) = l_R(\omega R_{i-1}/\omega R_i)$$

and we call *type sequence* of R the sequence $[r_1, \ldots, r_n]$.

We need in the sequel the following facts (see [1]):

(1) $r := r_1$ is the *Cohen-Macaulay type* of R.

(2) $1 \leq r_i \leq r_1$ for every $i \geq 1$.

(3) $\delta = \sum_1^n r_i$, and $2\delta - c = l_R(\omega/R) = \sum_1^n (r_i - 1)$.

It follows immediately that $r - 1 \leq 2\delta - c$.

(4) If $s_i \in v(\theta_D)$, then $r_{i+1} = 1$ (see [14], Prop.3.4).

(5) $r_i = 1$ for every $i > n$.

2.3
A ring R is called *almost Gorenstein* if it satisfies the equivalent conditions
(1) $\mathfrak{m} = \mathfrak{m}\,\omega$.
(2) $r - 1 = 2\delta - c$.
By the above property 2.2,(3), it is clear that R is *almost Gorenstein* if and only if the type sequence is $[r, 1, \dots, 1]$ and that *Gorenstein* means *almost Gorenstein* with $r = 1$.
A ring R is called *of maximal length* if it satisfies the equivalent conditions
(1) $r(c - \delta) = \delta$.
(2) the type sequence is constant $[r, r, \dots, r]$.

2.4
For any proper ideal I of R, we denote by $\overline{I} := I\overline{R} \cap R$ the *integral closure* of I. Easily we can see that

$$I \subseteq I^{**} \subseteq \omega I = \omega I^{**}.$$

In fact, $I^{**} = R : (R : I) \subseteq \omega : (R : I) = \omega I$ and $l_R(\omega I^{**}/\omega I) = l_R(I^*/I^{***}) = 0$. Hence $I^{**} \subseteq \overline{I}$ and $e(I^{**}) = e(I)$. We note also that the condition $\omega \subseteq I : I$, i.e. $\omega I = I$, implies that $I = I^{**}$.

2.5
For any fractional ideal I we denote by γ_I the biggest $\overline{R}$-ideal contained in I and by c_I the multiplicity of γ_I. Namely:

$$\gamma_I := t^{c_I}\overline{R} \subseteq I \quad \text{with} \quad c_I - 1 \notin v(I), \quad R : \gamma_I = t^{c - c_I}\overline{R}, \quad v(R : \gamma_I) = \mathbb{Z}_{\geq c - c_I}.$$

Assume now that $I \subseteq R$ and let $n_I := l_R(R/\gamma_I) = c_I - \delta \geq n$. Then

(1) $\gamma_I \subseteq \gamma$ and the inclusion $\gamma \subseteq I$ implies that $\gamma_I = \gamma$.

(2) $\sum_{i=1}^{n_I} r_i = l_R(R : \gamma_I/R) = c_I - c + \delta$ and
$\sum_{h=1}^{n_I} (r_h - 1) = 2\delta - c$.

(3) From the square

$$\begin{array}{ccc} R & \subseteq & \overline{R} \\ \cap\!| & & \cap\!| \\ I^* & \subseteq & R : \gamma_I \end{array}$$

and the above item we get

$$l_R(I^*/R) = \sum_{i=1}^{n_I} r_i - l_R(R : \gamma_I/I^*)$$

3 Invariants a(I) and b(I).

For any proper ideal I of R, we define the two invariants

$a(I) := l_R(I^*/R) - l_R(R/I)$

$$b(I) := r l_R(R/I) - l_R(I^*/R),$$

in particular: $a(\gamma) = 2\delta - c, \quad b(\gamma) = r(c-\delta) - \delta, \quad a(\mathfrak{m}) = r-1, \quad b(\mathfrak{m}) = 0.$

The aim of the section is to express these invariants in terms of the type sequence of R. The particular description given in Theorem 3.10 allows us to get bounds and vanishing conditions, improving results of several authors.

First we collect some remarks concerning $a(I)$ and $b(I)$.

Remark 3.1 Let I be a proper ideal of R. Then:

(1) $a(I) + b(I) = (r-1)\, l_R(R/I)$.

(2) $a(I) = a(\gamma) - l_R(\omega I/I) \leq a(\gamma)$.

This easy computation yields immediately that:

(a) $a(I) = 0$ for every ideal $I \iff R$ is Gorenstein

(b) $a(\mathfrak{m}) = a(\gamma) \iff R$ is almost Gorenstein

(c) I *canonical*, i.e. $I \simeq \omega, \implies a(I) = a(\gamma) - l_R(R/\theta_D)$.

For a discussion about the invariant $\sigma := a(\gamma) - l_R(R/\theta_D)$ see [14], 3.5, where we found examples with $\sigma < 0$.

(3) $b(I) \geq 0$.

This fact follows by applying with $M = N = R$ the Jäger's inequality:

$$l_R(M : I/M : N) \leq l_R(M : \mathfrak{m}/M) l_R(N/I)$$

which holds for every fractional ideals $M,\ N,\ I,$ such that $I \subseteq N$ (see [10], Satz 2).

(4) If $J \subseteq I$, we have:

(a) $a(J) - a(I) = l_R(J^*/I^*) - l_R(I/J)$.

(b) $b(J) - b(I) = r l_R(I/J) - l_R(J^*/I^*) \geq 0$.

Assertion (a) is easy to check and (b) follows directly from (a) by means of (1). The positivity of $b(J) - b(I)$ is again a consequence of the Jager's result. We note in particular that:

(c) $a(I) = a(I^{**}) - l_R(I^{**}/I)$.

(d) $b(I) = 0$ for every ideal I containing γ if and only if R is a ring *of maximal length*.

(5) By definition $\sum_{h=1}^{i} r_h = l_R(R : R_i/R)$. Then:

(a) $a(R_i) = \sum_{h=1}^{i}(r_h - 1)$, in particular $a(R_i) = 2\delta - c$ for every $i \geq n$.

(b) $b(R_i) = \sum_{h=1}^{i}(r - r_h)$, in particular

for $i \geq n$, i.e. $R_i = t^{c+p}\overline{R}$, $p \geq 0$, we get $b(R_i) = b(\gamma) + p(r-1)$.

(6) If R is Arf, i.e. $l_R(R : R_i/R) = s_i - i$ for every $1 \leq i \leq n$ (see [5], Proposition 1.15), then

$$a(I) \leq (r-1)l_R(R/I) - (i_0 s_1 - s_{i_0})$$

where s_{i_0} is the multiplicity of I.

In fact, the hypothesis R Arf implies that $a(R_i) = s_i - 2i$, $b(R_i) = is_1 - s_i$. Applying the second formula of (4) to the ideals $I \subseteq R_{i_0} = \overline{I}$ we obtain $b(I) \geq i_0 s_1 - s_{i_0}$, hence the thesis by (1).

We introduce now another notation.

Notation 3.2 *We associate to any proper ideal I the numerical set V^I depending on the valuations of I^{**}*

$$V^I := \{h+1 \mid h \in \mathbb{N} \text{ and } s_h \in v(I^{**})\}.$$

The r_is of the *type sequence*, with $i \in V^I$, will be useful in our computations.

Remark 3.3 *Let, as usual, $n_I = c_I - \delta$. Then:*

$$\#V^I_{\leq n_I} = l_R(I^{**}/\gamma_I) \quad \text{and} \quad \#V^I_{\leq n} = l_R(I^{**} + \gamma/\gamma).$$

The basic idea for the next theorem comes from 2.1.(5), which establishes a duality between the valuations of ωI and those of I^*.

Theorem 3.4 *For any proper ideal I we have:*

1. $l_R(I^{**}/\gamma_I) \leq \displaystyle\sum_{h \leq n_I,\ h \in V^I} r_h \leq l_R(R : \gamma_I/I^*)$.

2. $l_R(I^*/R) \leq \displaystyle\sum_{h \notin V^I} r_h = l_R(R/I^{**}) + \sum_{h \leq n,\ h \notin V^I} (r_h - 1)$.

<u>Proof</u>. The proof is substantially the same as in [15], Proposition 4.2; some changes are due to the fact that now we don't assume that I is a reflexive ideal containing γ.

(1) The first inequality is true by 3.3, since $r_h \geq 1$ for each h.
For the last one let h be an integer, $1 \leq h \leq n_I$. If $x_{h-1} \in I^{**}$ is such that $v(x_{h-1}) = s_{h-1} < c_I$, then by definition

$$r_h = l_R(\omega R_{h-1}/\omega R_h) = l_R(x_{h-1}\omega + \omega R_h/\omega R_h) = \#\{v(x_{h-1}\omega + \omega R_h) \setminus v(\omega R_h)\}.$$

Since $v(x_{h-1}\omega) \subseteq v(\omega I^{**}) = v(\omega I)$, by virtue of 2.1,(5) the assignement $y \to c - 1 - y$ defines an injective map

$$\bigcup_{h \in V^I_{\leq n_I}} \{v(x_{h-1}\omega + \omega R_h) \setminus v(\omega R_h)\} \longrightarrow \mathbb{Z}_{\geq c - c_I} \setminus v(I^*).$$

The conclusion $\sum_{h \in V^I_{\leq n_I}} r_h \leq l_R(R : \gamma_I/I^*)$ follows, because the sets $\{v(x_{h-1}\omega + \omega R_h) \setminus v(\omega R_h)\}$, $h \in V^I_{\leq n_I}$, are disjoint by construction and because $\mathbb{Z}_{\geq c - c_I} = v(R : \gamma_I)$.

(2) The last inequality in (1) combined with 2.5 (3) gives:

$l_R(I^*/R) \leq \sum_{h=1}^{n_I} r_h - \sum_{h \in V^I_{\leq n_I}} r_h = \sum_{h \notin V^I} r_h =$

$= l_R(R/I^{**}) + \sum_{h \notin V^I} (r_h - 1).$

The thesis is now immediate since $r_h = 1$ for all $h > n$. $\diamond$

Corollary 3.5 *For any proper ideal I we have:*

$$l_R(\omega I/I) \geq \sum_{h \in V^I} (r_h - 1)$$

Proof. By 3.1, (4) and part (2) of the theorem, we obtain

$$a(I) \leq a(I^{**}) \leq \sum_{h \notin V^I} (r_h - 1).$$

Using 3.1, (3), we conclude that:

$$l_R(\omega I/I) = 2\delta - c - a(I) \geq \sum_{h=1}^{n} (r_h - 1) - \sum_{h \notin V^I} (r_h - 1),$$

which is the thesis. $\diamond$

The last inequality in Theorem 3.4, (1) leads to introduce the following non-negative invariant.

Definition 3.6 *For any proper ideal I we define*

$$d(I) := l_R(R : \gamma_I/I^*) - \sum_{h \leq n_I,\ h \in V^I} r_h.$$

It is clear that:

1. $d(I) = d(uI)$ for every unit $u \in \overline{R}$;
2. $d(I) \geq 0$, by 3.4;
3. $l_R(R : \gamma_I/I^*) - r l_R(I^{**}/\gamma_I) \leq d(I) \leq l_R(R : \gamma_I/I^*) - l_R(I^{**}/\gamma_I)$ and the minimal value is achieved in a ring of maximal length.

Corollary 3.7 *Let I be a proper ideal. Then*

1. $l_R(I/\gamma_I) \leq l_R(R : \gamma_I/I^*)$.
2. *Equality holds in (1) $\iff$ I is reflexive, $d(I)=0$, $r_h = 1$ $\forall\, h \in V^I_{\leq n_I}$.*

Proposition 3.8 *For any proper ideal I we have:*

1. $d(I) = l_R(\omega I/I^{**}) - \sum_{h \in V^I} (r_h - 1)$.
2. $d(I^{**}) = d(I)$.
3. *If* $I \subseteq \theta_D$, *then* $d(I) = l_R(\omega I/I^{**})$.

4. *If* $\omega \subseteq I : I$, *then* $d(I) = 0$.

5. *Let* $i_o \in \mathbb{N}$ *be the integer such that* $e(I) = s_{i_0}$. *Then*
$d(I) = \sum_{h > i_0,\ h \notin V^I} r_h \ - \ l_R(I^*/R^*_{i_0})$.

6. *If* I *is integrally closed, then* $d(I) = 0$.

7. *If* R *is almost Gorenstein, then* $d(I) = 0$.

Proof.

(1) By (2) of 2.1 $l_R(R : \gamma_I/I^*) = l_R(\omega I/\gamma_I)$. Thus:
$d(I) = l_R(\omega I/\gamma_I) - \sum_{h \in V^I_{\leq n_I}} r_h = l_R(\omega I/I^{**}) - (\sum_{h \in V^I_{\leq n_I}} r_h - l_R(I^{**}/\gamma_I)) =$
$l_R(\omega I/I^{**}) - \sum_{h \in V^I} (r_h - 1)$.

(2) It is a consequence of item (1), in view of the fact that $\omega I = \omega I^{**}$ by 2.4 and $V^I = V^{I^{**}}$ by definition.

(3) It follows from (1) in view of 2.2 (4).

(4) The inclusion $\omega \subseteq I : I$ implies that $\omega I = I^{**}$, hence the thesis by (3).

(5) After writing $l_R(R : \gamma_I/I^*) = l_R(R : \gamma_I/R^*_{i_0}) - l_R(I^*/R^*_{i_0})$, the thesis is clear since
$$l_R(R : \gamma_I/R^*_{i_0}) = \sum_{i_0 < h \leq n_I} r_h.$$

(6) It follows from the above item, because $I = R_{i_0}$.

(7) We prove that $\omega I = I^{**}$. As observed in 2.4, the inclusion $I^{**} \subseteq \omega I$ always holds. Now $\omega I(R : I) \subseteq \omega \mathfrak{m} = \mathfrak{m}$. Thus $\omega I \subseteq I^{**}$. The thesis comes from (1) combined with the fact that $d(I) \geq 0$. $\diamond$

The next theorem extends to any birational overring S of R the formulas proved in [15] in the case of the blowing-up Λ of R along a proper ideal. We remark also that for $S = \overline{R}$ the first inequality $l_R(S/R) \leq r\ l_R(R/R : S)$ becomes the well-known relation $\delta \leq r(c - \delta)$.

Theorem 3.9 *Let* S *be an* R*-overring,* $R \subseteq S \subseteq \overline{R}$ *and let* $I := R : S$ *be its conductor ideal. Let* $i_o \in \mathbb{N}$ *denote the integer such that* $e(I) = s_{i_0}$. *Then:*

$$l_R(S/R) = \sum_{h \notin V^I} r_h - l_R(S^{**}/S) - d(I) \leq r\ l_R(R/I)$$
$$l_R(S/R) = \sum_{h \leq i_0} r_h - l_R(S^{**}/S) + l_R(S^{**}/R^*_{i_0})$$

Proof. Since the hypothesis $R \subseteq S \subseteq \overline{R}$ ensures that $\gamma_I = \gamma$, the proof of Theorem 4.4 of [15] works in the general case and we may omit the proof. $\diamond$

From Theorem 3.4 we deduce now the following two formulas which connect the invariants $a(I)$, $b(I)$ with the type-sequence.

Theorem 3.10 *For any proper ideal I of R we have:*

1. $a(I) = \sum_{h \notin V^I} (r_h - 1) - l_R(I^{**}/I) - d(I)$.
2. $b(I) = \sum_{h \notin V^I} (r - r_h) + r l_R(I^{**}/I) + d(I)$.

Proof.

(1) By 2.5, (3):

$$a(I) + d(I) + l_R(I^{**}/I) =$$
$$= l_R(I^*/R) - l_R(R/I) + l_R(R : \gamma_I/I^*) - \sum_{h \in V^I_{\leq n_I}} r_h + l_R(I^{**}/I) =$$
$$= \sum_{h=1}^{n_I} r_h - \sum_{h \in V^I_{\leq n_I}} r_h - l_R(R/I^{**}) =$$
$$= \sum_{h \notin V^I} (r_h - 1).$$

(2) It follows from (1), since $a(I) + b(I) = (r-1) l_R(R/I)$.

We get immediately interesting lower and upper bounds.

Corollary 3.11 *The following inequalities hold:*

1. $a(I) \leq (r-1) l_R(R/(I^{**} + \theta_D)) - l_R(I^{**}/I)$.
   $a(I) \geq r - 1 - l_R(I^{**}/I) - d(I)$.
2. $b(I) \leq (r-1)(l_R(R/I) - 1) + l_R(I^{**}/I) + d(I)$.
   $b(I) \geq (r-1) l_R((I^{**} + \theta_D)/I) + l_R(I^{**}/I)$.

Proof. First recall the positivity of $d(I)$ and some properties of type sequences:
(i) $r_h \leq r$ for every $h = 1, ..., n$;
(ii) $r_h = 1$ for every $h > n$ and for every h such that $s_{h-1} \in v(\theta_D)$.
Then derive assertions of part (1) from the first formula of the theorem.
Since $a(I) + b(I) = (r-1)\, l_R(R/I)$, (2) follows easily from (1). $\diamond$

The first statement in item (1) of the corollary improves the inequality $a(I) \leq (r-1) l_R(R/I)$ obtained by Jäger in [10], Korollar 3, (2).

The two statements in item (2) generalize to any ideal I the upper bound $b(\gamma) \leq (r-1)[l_R(R/\gamma) - 1]$ and the lower bound $b(\gamma) \geq l_R(\theta_D/\gamma)(r-1)$, already known for the conductor ideal (see, respectively, [5], Proposition 2.1 and [14], Theorem 3.7).

The second statement in item (1) provides a sufficient condition for the positivity of $a(I)$. Using 2.4 and 3.8, we have immediately that

Corollary 3.12 *If I satisfies the condition $\omega \subseteq I : I$, then $a(I) \geq r - 1 \geq 0$.*

Another direct consequence of 3.10 is the following.

Corollary 3.13

1. $b(I) \geq r\ l_R(I^{**}/I) \geq 0$.
2. *(Vanishing condition for* $b(I)$*).*

   $b(I) = 0 \Longleftrightarrow I = I^{**},\ r_h = r\ \ \forall h \notin V^I$ *and* $\displaystyle\sum_{h \in V^I,\ h \leq n_I} r_h = l_R(R : \gamma_I/I^*).\diamond$

Finally we obtain a characterization of the *almost Gorenstein* property in terms of the invariant $a(I)$ (see next $1 \Longleftrightarrow 5$), which is just the analogue of a theorem stated by E. Matlis for Gorenstein rings (see [11], Theorem 13.1).

Theorem 3.14 *Here "ideal" means " fractional ideal". The following facts are equivalent:*

1. *R is almost Gorenstein.*
2. $\omega I = I^{**}$ *for every non-principal ideal I.*
3. $l_R(I/J) = l_R(J^*/I^*)$ *for every reflexive ideals I, J,* $J \subseteq I$.
4. $l_R(I/\gamma_I) = l_R(R : \gamma_I/I^*)$ *for every reflexive ideal I.*
5. $a(I) = (r-1) - l_R(I^{**}/I)$ *for every non-principal ideal $I \subseteq R$.*
6. $r - 1 = 2\delta - c$.
7. $\mathfrak{m}\omega = \mathfrak{m}$.

Proof.

(1) $\Longrightarrow$ (2) As observed in 2.4, the inclusion $I^{**} \subseteq \omega I$ always holds. Now $\omega I(R : I) \subseteq \omega\mathfrak{m} = \mathfrak{m}$. Thus $\omega I \subseteq I^{**}$.

(2) $\Longrightarrow$ (3) By 2.1: $l_R(J^*/I^*) = l_R(\omega I/\omega J) = l_R(I/J)$.

(3) $\Longrightarrow$ (4) Take $J = \gamma_I$.

(4) $\Longrightarrow$ (6) Take $I = \mathfrak{m}$.

(1) $\Longrightarrow$ (5) This implication follows from 3.10, because in the almost Gorenstein case $r_h = 1$ for all $h \neq 1$ and $d(I) = 0$ by 3.8, (7).

(5) $\Longrightarrow$ (6) Take $I = \gamma$.

(6) $\Longleftrightarrow$ (7) $\Longleftrightarrow$ (1) These equivalences are well-known.

4 The special case of γ.

The description of the invariant $b := b(\gamma)$ in terms of type sequence given in Theorem 3.10

$$b = \sum_{h=1}^{n}(r - r_h)$$

allows us to complete the classification of all analitically irreducible local rings having $b \leq r$. Some of the results contained in this section are already present in the literature (see [6], [7], [8]).

From now on we shall denote by $x \in \mathfrak{m}$ an element such that $v(x) = e$, in other words xR is a minimal reduction of $\mathfrak{m}$.

Lemma 4.1
Let $z := min\{y \in v(R) \mid y \geq c - e\}$ *and let* $B := \{h \in [1, n] \mid z < s_h \leq c\}$.

1. $\#B = l_R(\gamma :_R \mathfrak{m}/\gamma) = l_R(R/\gamma + xR) \geq e - r \geq 1$.
2. $\sum_{h \in B} r_h \leq e - 1$.

<u>Proof</u> First of all we observe that, called $i_0 := min(B)$, we have by definition $z = s_{i_0 - 1}$ and $B = [i_0, n]$.

(1) Obviously we have that

$$v(\gamma :_R \mathfrak{m}) \setminus v(\gamma) = \{s_i \in v(R) \mid \quad c - e \leq s_i < c\}.$$

Clearly this set is in 1-1 correspondence with the set

$$\{i \mid z \leq s_i < c\} = [i_0 - 1, n - 1],$$

so the first assertion of (1) is proved.

It is easy to check that $\qquad x(\gamma :_R \mathfrak{m}) = xR \cap \gamma$.

Hence $\quad l_R\big(xR/x(\gamma :_R \mathfrak{m})\big) = l_R(xR/xR \cap \gamma) = l_R(\gamma + xR/\gamma)$ and to prove the second equality it suffices to consider the following inclusions

$$\begin{array}{ccc} \gamma :_R \mathfrak{m} & \subseteq & R \\ \cup| & & \cup| \\ \gamma & \subseteq & \gamma + xR \end{array}$$

Finally, since $(\gamma + xR)\mathfrak{m} \subseteq xR$, we obtain $(\gamma + x\mathfrak{m}) \subseteq xR : \mathfrak{m}$, hence

$l_R(\gamma + xR/xR) \leq r$ and

$l_R(R/\gamma + xR) = l_R(R/xR) - l_R(\gamma + xR/xR) \geq e - r$.

(2) Since $c - 1 \notin v(\omega)$ by (4) of 2.1, $\quad v(\omega R_{i_0 - 1})_{<c} \subseteq [c - e, c - 2]$. Thus: $\sum_{h \in B} r_h = l_R(\omega R_{i_0 - 1}/\gamma) \leq e - 1$. $\diamond$

Theorem 4.2 *Let* $A := \{1, ..., n\} \setminus B$. *The following inequalities hold:*

1. $b + e - 1 \geq b + \sum_{h \in B} r_h = \sum_{h \in A}(r - r_h) + r l_R(R/\gamma + xR)$.
2. $b \geq (r - 1)(e - r - 1) + \sum_{h \in A}(r - r_h)$.

Proof.

(1) We use the description of b in terms of type sequence given in 3.10.
$b = \sum_{h=1}^{n}(r - r_h) = \sum_{h\in A}(r - r_h) + \sum_{h\in B}(r - r_h) =$
$= \sum_{h\in A}(r - r_h) + r l_R(R/\gamma + xR) - \sum_{h\in B} r_h.$

(2) Since $l_R(R/\gamma + xR) \geq e - r$, by substituting in item (1) we get
$b \geq \sum_{h\in A}(r - r_h) + r(e - r) - (e - 1)$, which is our thesis. $\diamond$

Notation 4.3 We denote by

- p *the integer such that* $c - e \leq pe < c$ $(p \geq 1)$,

and by g the number of gaps of $v(R)$ in the interval (pe, c) :

- $g = \# \, \mathbb{N}_{\geq pe} \setminus v(R)$, $(1 \leq g \leq e - 1)$.

Formula 1 of Theorem 4.2 involves the length $l_R(R/\gamma + xR)$. For the proof of Theorem 4.7 we need next two lemmas, which describe in detail the cases $l_R(R/\gamma + xR) = 1, 2$.

Lemma 4.4 *The following facts are equivalent:*

1. $l_R(R/\gamma + xR) = 1$.
2. $v(R) = \{0, e, .., pe, c \rightarrow\}$.
3. $ts(R) = [e - 1, ..., e - 1, r_n]$.

If R satisfies these equivalent conditions, then R is a quasi-homogeneous singularity with
$\delta = c - p - 1$, $b = e(p + 1) - c \leq r - 1$, $r = e - 1$, $r_n = e - 1 - b$.

Proof. (1) $\iff$ (2) is immediate, and also the fact that R is a quasi homogeneous singularity, with $r = e - 1$ by (1) of 4.1. To prove (2) $\Longrightarrow$ (3), note that

$$\sum_{h=1}^{n-1} r_h = l_R(R : R_{n-1}/R) = l_R(x^{-p}R \cap \overline{R}/R) = ep - p = r(n - 1)$$

Hence $r_h = r$ for each $h \in [1, n - 1]$. Since $b = \sum_{h=1}^{n}(r - r_h)$ we get $r_n = r - b$. Therefore, $b < r$ and $ts(R) = [e - 1, ..., e - 1, e - 1 - b]$.
(3) $\Longrightarrow$ (2) follows, since for each $h \in [1, n - 1]$ the hypothesis $r_h = e - 1$ implies that $s_h = he$ (see [14], Proposition 4.9).

Lemma 4.5 *Assume that* $l_R(R/\gamma + xR) = 2$. *Then* $e - 2 \leq r \leq e - 1$ *and there are two possibilities for* $v(R)$:

(A) $v(R) = \{0, e, 2e, ..., ke, y, (k+1)e, y+e, ..., (p-1)e, y+(p-k-1)e, pe, c, \rightarrow\}$

with $p > k \geq 1$, $c \leq (p+1)e$, $y + (p - k)e \geq c$, $c - \delta = 2p + 1 - k$.

(B) $v(R) = \{0, e, 2e, ..., ke, y, (k+1)e, y+e, ..., pe, y+(p-k)e, c, \rightarrow\}$

$$\text{with} \quad p \geq k \geq 1, \ c \leq (p+1)e, \ \ y+(p-k)e < c, \quad c-\delta = 2p+2-k.$$

In both cases we have:

$\delta = p(e-1) - (p-k) + g = p(e-2) + k + g,$

$b + g = r(c-\delta) - p(e-2) - k$ *and* $1 \leq g \leq e-2.$

Moreover:

1. *If* $r = e-1$, *then* $b \geq r+1$ *and* $\left[\begin{array}{ll} \text{case (A)} & b+g = (p-k+1)e-1 \\ \text{case (B)} & b+g = (p-k+2)e-2 \end{array} \right.$

2. *If* $r = e-2$, *then* $p \geq 2k-1$

   and $\left[\begin{array}{ll} \text{case (A)} & b+g = (p-k+1)(e-2) - k \geq k(e-3) \\ \text{case (B)} & b+g = (p-k+2)(e-2) - k > k(e-3). \end{array} \right.$

Proof. The fact that $e-2 \leq r \leq e-1$ follows immediately from 4.1.(1).

(1) In case (A)
$b+g = (e-1)(2p+1-k) - p(e-2) - k = (p-k+1)e-1.$
Then the inequality $g \leq e-2$ leads to $b \geq r+2$.
In case (B)
$b+g = (e-1)(2p+2-k) - p(e-2) - k = (p-k+2)e-2$
and the same inequality leads to $b \geq r+1$.

(2) It suffices to prove that $2y < c+e$; in fact from this we can deduce that

$$2ke < 2y < c+e \leq (p+2)e, \text{ hence } p > 2k-2.$$

If $2y \geq c+e$, then by considering the structure of $v(R)$ we can easily see that $\mathfrak{m}^2 \subseteq t^e\mathfrak{m}$. Thus, $\mathfrak{m} = t^e(R : \mathfrak{m}) \subseteq R \subseteq R : \mathfrak{m}$, contradicting the assumption $r = e-2$. $\diamond$

Corollary 4.6 *Assume that* $b < q(r-1)$, $q \geq 1$, *then*

$$e - r \leq \ l_R(R/\gamma + xR) \leq q.$$

In particular

1. $0 \ \leq b < \ r-1 \implies r = e-1$ *and* $l_R(R/\gamma + xR) = 1.$
2. $r-1 < b < 2r-2 \implies e-2 \leq r \leq e-1$ *and* $l_R(R/\gamma + xR) = 2.$

Proof. Item (2) of 4.2 implies that $(r-1)(e-r-1-q)<0$, so $e-1-q<r$ and item (1) gives $rl_R(R/\gamma+xR)<e-1+q(r-1)<r(q+1)$; hence the thesis using also 4.1 (1).
(a) is the case $q=1$, (b) is the case $q=2$, with the further assumtion $b>r-1$. It suffices to recall that by 4.4 $l_R(R/\gamma+xR)=1 \Longrightarrow b\leq r-1$.

From these technical observations and Theorem 4.2 we deduce the statements of the next theorem, which are partially already known (see [4], [7], [8], [6]). Nevertheless, they give a complete classification of all analitically irreducible local rings having $b\leq r$.

We shall consider separately the cases: 1) $b<r-1$; 2) $b=r-1$; 3) $b=r$.

Theorem 4.7 *Suppose R not Gorenstein.*

1. *The following facts are equivalent:*
   (a) $b<r-1$
   (b) $v(R)=\{0,e,..,pe,c\rightarrow\}$ *with* $pe+2<c\leq(p+1)e$
   (c) $ts(R)=[e-1,e-1,...,e-1,r_n]$, $r_n>1$.
   If these conditions hold, then
   $$l_R(R/\gamma+xR)=1,\quad c=(p+1)e-b,\quad r=e-1,\ r_n=e-1-b.$$
2. $$b=r-1\Longrightarrow\begin{cases} r=e-1\\ \textit{or}\\ r=e-2\end{cases}$$
   1^{st} *case) The following facts are equivalent:*
   (a) $b=r-1=e-2$
   (b) $v(R)=\{0,e,..,pe,c\rightarrow\}$ *with* $c=pe+2$
   (c) $ts(R)=[e-1,e-1,...,e-1,1]$.

   If these conditions hold, then $l_R(R/\gamma+xR)=1$.
   2^{nd} *case) The following facts are equivalent:*
   (d) $b=r-1=e-3$
   (e) either $v(R)=\{0,e,2e-1,2e,3e-1\rightarrow\}$
   or $v(R)=\{0,e,y,2e\rightarrow\}$ *with* $e<y\leq e+\frac{e-1}{2}$
   (f) either $ts(R)=[e-2,e-2,r_3,r_4]$ *with* $r_3+r_4=e-1$
   or $ts(R)=[e-2,r_2,r_3]$ *with* $r_2+r_3=e-1$.

   If these conditions hold, then $l_R(R/\gamma+xR)=2$.
3. $$b=r\Longrightarrow\begin{cases} (g)\quad r=e-2,\ l_R(R/\gamma+xR)=2\\ \textit{or}\\ (j)\quad r=2,\ e=5,\ l_R(R/\gamma+xR)=3.\end{cases}$$
   In case (g), $v(R)$ *is one of the following sets*

$\{0, 4, 8, 9, 12, 13, 16 \rightarrow\}$;
$\{0, 4, 8, 11, 12, 15, 16, 19 \rightarrow\}$;
$\{0, e, 2e-2, 2e, 3e-2 \rightarrow\}$, *with* $e \geq 4$;
$\{0, e, e+z, 2e-1 \rightarrow\}$, *with* $0 < z \leq \dfrac{e-2}{2}$, $e \geq 4$.

In case (j), $v(R)$ *is one of the following sets (see [6], Rem. 2.7)*

$\{0, 5, 6, 7, 10 \rightarrow\}$;
$\{0, 5, 6, 8, 10 \rightarrow\}$;
$\{0, 5, 8, 9, 10, 13 \rightarrow\}$

Proof

(1) $(a) \Longrightarrow (b)$. If $b < r-1$, then by 4.6,2, $l_R(R/\gamma + xR) = 1$.
By Lemma 4.4 $v(R) = \{0, e, 2e, ..., pe, c, \rightarrow\}$ with $(p+1)e \geq c$. Then $b = (p+1)e - c < e-2$ implies that $pe + 2 < c$.

$(b) \Longrightarrow (c)$. The hypothesis $pe+2 < c$ gives $b < r-1$. Then $r_n = r-b > 1$.

$(c) \Longrightarrow (a)$. We have $b = r - r_n$, hence the thesis.

(2) By substituting $b = r-1$ in Formula 2 of 4.2, we get $(r-1)(e-r-2) \leq 0$. Two cases are possible: $r = e-1$ or $r = e-2$ and $\sum_{h \in A}(r - r_h) = 0$.

First case.

$(a) \Longrightarrow (b)$. As in (1) one gets $l_R(R/\gamma + xR) = 1$. Then $v(R) = \{0, e, .., pe, c \rightarrow\}$ and $b = (p+1)e - c = e-2$, hence $c = pe+2$.

$(b) \Longrightarrow (c)$. See Lemma 4.4.

$(c) \Longrightarrow (a)$. In fact, $b = r - r_n = r-1$.

Second case.

$(d) \Longrightarrow (e)$. If $b = r-1$ and $r = e-2$, then by (2) of 4.2 we have $\sum_{h \in A}(r - r_h) = 0$ and from item (1) of 4.2 we obtain $l_R(R/\gamma + xR) = 2$. It follows that $t.s(R) = [e-2, ..., e-2, r_{n-1}, r_n]$.

We have to consider the two cases of Lemma 4.5.

In case (A) with $b = e-3$, from the inequality $g \geq k(e-3) - b$ we get

$$e - 2 \geq g \geq (k-1)(e-3)$$

Three possibilities occur:

1) $k = 1$. Then $p = 2$, $g = e-2$, $c = 3e-1$, $y = 2e-1$. In conclusion

$$v(R) = \{0, e, 2e-1, 2e, 3e-1 \rightarrow\}.$$

2) $k = 2$. Then $p = 3$, $g = e-3$, $c = 4e-2$, $y = 3e-2$, so $2y > c+e$, absurd (see (2) in the proof of 4.5).

3) $e = 4$, $k = 3$. Then $p = 5$, $g = 2$, $c = 23$, $y = 15$, as above impossible since $2y > c+e$.

In case (B) with $b = e-3$, since $g \geq k(e-3) + e - 2 - b$, we obtain

$$e - 2 \geq g \geq k(e-3) + 1$$

The only possibility is $k = 1$. Then we get $p = 1$, $g = e - 2$ and $v(R) = \{0, e, y, 2e \rightarrow\}$ with $e < y \leq e + (e-1)/2$.

$(e) \Longrightarrow (f)$. Let R_0 be the monomial ring such that $v(R_0) = v(R) = \{0, e, 2e-1, 2e, 3e-1 \rightarrow\}$. Then $r(R) \leq r(R_0) = e-2$. Since $l_R(R/\gamma + xR) = 2$, we have by item (2) of 4.5 $r(R) \geq e-2$. Then $r(R) = e-2$. We easily compute $b = (c-\delta)r - \delta = e-3$. Substituting in item (2) of 4.2 we obtain $\sum_{h \in A}(r - r_h) = 0$, hence $r_2 = e-2$ and $r_3 + r_4 = e-1$. The same reasoning holds for $v(R) = \{0, e, y, 2e \rightarrow\}$.

$(f) \Longrightarrow (d)$. It suffices to recall that $b = \sum_{h=1}^{n}(r - r_h)$.

(3) Assume $b = r$. From item (2) of 4.2 it follows that $(r-1)(e-r-2) \leq 1$, then using also 4.2,(1) we argue that either $r = 2$ and $e \leq 5$, or $l_R(R/\gamma + xR) = 2$ and $r \geq e-2$.
Since the cases $r = 2$, $e = 3$ and $l_R(R/\gamma + xR) = 2$, $r = e-1$ are impossible by Lemma 4.5, the first assertion is proved.

Case (g): $l_R(R/\gamma + xR) = 2$ and $b = r = e-2$.
We proceed analogously to the proof of (2).
In case (A) we have

$$e - 2 \geq g = (p-k)(e-2) - k \geq (k-1)(e-3) - 1$$

This gives the following possibilities:

1) $k = 1$. Then $p = 2$ $g = e-3$, $c = 3e-2$, $y = 2e-2$. Hence

$$v(R) = \{0, e, 2e-2, 2e, 3e-2 \rightarrow\}, \quad e \geq 4.$$

2) $k = 2$.

i) $k = 2$, $p = 4 = e$. Then $g = 2$, $c - \delta = 7$, $\delta = 12$, $c = 19$, $y = 11$,

$$v(R) = \{0, 4, 8, 11, 12, 15, 16, 19 \rightarrow\}$$

ii) $k = 2$, $p = 3$. Then $g = e-4$, $c = 4e-3$, $y \geq 3e-3 \Longrightarrow 2y > c+e$ impossible.

3) $k = 3$.

i) $k = 3$, $e = 5$. Then $p = 5$, $g = 3, c = 29$, $y = 19 \Longrightarrow 2y > c+e$ impossible.

ii) $k = 3$, $e = 4$. Then $p = 5$, $g = 1, c = 22$, $y = 14 \Longrightarrow 2y > c+e$, impossible.

4) $k = 4$, $e = 4$. Then $p = 7$, $g = 2$, $c = 31$, $y = 19 \Longrightarrow 2y > c+e$ impossible.

In case (B) we have

$$e - 2 \geq g = (p-k+1)(e-2) - k \geq k(e-3)$$

and the following possibilities:

1) $k = 1$. Then $p = 1$, $g = e-3$, $c = 2e-1$. Hence

$$v(R) = \{0, e, e+z, 2e-1 \rightarrow\} \text{ with } 0 < z \leq \frac{e-2}{2}, \quad e \geq 4$$

2) $k = 2$, $e = 4$. Then $p = 3$, $g = 2$, $v(R) = \{0, 4, 8, 11, 12, 15, 16 \rightarrow\}$.

Case (j) is treated in [6]. $\diamond$

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